Symplectic geometry

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1 Motivation for symplectic geometry

Smooth manifolds in this lecture will be considered Hausdorff and second countable. Unless otherwise stated, we will assume manifolds to have constant dimension.

Remark 1.1. *Almost everything stays true for manifolds with connected components of varying dimension. Sometimes 'boundedness' of the dimension is useful to obtain certain 'finiteness' results. For instance, we will have statements of the type: any differential form can be written as a finite sum of wedge products of one-forms, which is not true for manifolds like* $\bigsqcup_i \mathbb{R}^i$.

We recall the following facts about smooth manifolds:

- For any smooth manifold M, the tangent bundle TM is a vector bundle.[1]
- The space of smooth sections $\Gamma(TM)$ of TM is denoted by $\mathfrak{X}(M)$. Its elements are called vector fields.
- There is a canonical isomorphism $\mathfrak{X}(M) \cong \mathrm{Der}(C^\infty(M))$, where the latter is the space of derivations of the $\mathbb{R}$-algebra $C^\infty(M)$, i.e. the space of linear maps $D : C^\infty(M) \to C^\infty(M)$ satisfying $D(f \cdot g) = D(f) \cdot g + f \cdot D(g)$.
- Vector fields carry a *commutator bracket*, which is skew-symmetric and satisfies the Jacobi identity.

1.1 From gradients to Poisson structures (GEOMETRY)

Let M be a smooth manifold. Recall that for any smooth function $f \in C^\infty(M)$, there is a natural section of the cotangent bundle $df \in \Gamma(T^*M)$, given by the differential. In coordinates x_i it is given by $df = \sum_i \frac{\partial f}{\partial x_i} dx_i$. Note

[1]A vector bundle of rank k over M is a smooth manifold E with a projection $\pi_E : E \to M$ and fiberwise vector space structures, such that for any $x \in M$ there is an open neighborhood $U_x \subset M$ and a diffeomorphism $E|_{U_x} := \pi_E^{-1}(U_x) \cong U_x \times \mathbb{R}^k$ intertwining the projections onto U_x and the fiberwise vector space structures.

that this is a linear form (in coordinates: a row vector). To obtain the gradient ∇f, which is a vector field (in coordinates : a column vector), we need to use an isomorphism $\rho : T^*M \to TM$, usually the musical isomorphism $\rho = g^\sharp$ induced by a Riemannian metric $g : TM \times_M TM \to \mathbb{R}$. The gradient is supposed to indicate the steepest direction for changes in f, and this is due to a property of the metric:

$$X(f) = df(X) = g(\nabla f, X)$$

For X of prescribed length, this expression will always be maximal when X is (an appropriate multiple) of ∇f.

Imagine now, that we want to do the opposite: Given a function f, we want to find directions in which f does not change. What is the property we might want from ρ? Let us write

$$\nabla^\rho f := \rho(df).$$

Then we can calculate

$$0 = \nabla^\rho f(f) = df(\nabla^\rho f) = df(\rho(df)).$$

So ρ should satisfy $\alpha(\rho(\alpha)) = 0$ for all $\alpha \in T^*M$. If we reinterpret $\rho : T^*M \to TM$ as a map $\pi : T^*M \times_M T^*M \to \mathbb{R}$[2] (and write ∇^π rather than ∇^ρ) this would mean that π is skew-symmetric. We can use all of this to define a skew-symmetric bracket

$$\{f, g\}^\pi := (\nabla^\pi f)(g) = -(\nabla^\pi g)(f)$$

on smooth functions. For this to be a useful construction, we would need this new bracket to be compatible with the Lie bracket of vector fields, i.e. satisfy the equality

$$\nabla^\pi \{f, g\}^\pi = [\nabla^\pi f, \nabla^\pi g] \tag{1}$$

Definition 1.2. *A* Poisson structure *on a smooth manifold M is a skew-symmetric map $\pi : T^*M \times_M T^*M \to \mathbb{R}$ such that for all pairs of smooth functions f, g the identity $\nabla^\pi\{f, g\}^\pi = [\nabla^\pi f, \nabla^\pi g]$ holds.*

Let us try to see what is happening in local coordinates. Assume $M = \mathbb{R}^n$ equipped with the standard frame of the tangent bundle $\partial_{x_1}, ..., \partial_{x_n}$ and the dual frame of the cotangent bundle $dx_1, ..., dx_n$. A smooth map $\pi : T^*M \times_M T^*M \to \mathbb{R}$ is then just given as by a collection of smooth functions

$$\pi(dx_i, dx_j) =: \pi_{i,j} \in C^\infty(M).$$

The ρ corresponding to π is then the map $\rho(dx_i) = \sum_j \pi_{i,j}\partial_{x_j}$ and the bracket becomes

$$\{f, g\}^\pi = \sum_{i,j} \pi_{i,j} \frac{\partial f}{\partial x_i} \frac{\partial g}{\partial x_j}.$$

The skew-symmetry simply means $\pi_{i,j} = -\pi_{j,i}$. The following Lemma will help us verify Equation (1):

Lemma 1.3. *If for all i, j, we have $\nabla^\pi\{x_i, x_j\}^\pi = [\nabla^\pi x_i, \nabla^\pi x_j]$, then Equation* (1) *holds for all f, g.*

Proof. Let us write $c^\pi(f, g)$ for $[\nabla^\pi f, \nabla^\pi g] - \nabla^\pi\{f, g\}^\pi$. Since ∇^π, $[\cdot, \cdot]$ and $\{\cdot, \cdot\}^\pi$ all depend only on the (value and) first derivatives of their arguments, the value of $c^\pi(f, g)$ at some point x can only depend on the values, first and second derivatives of f and g at that point. This means that if we show that $c^\pi(f, g)$ vanishes for f, g arbitrary polynomials of degree ≤ 2, then it will vanish on all pairs of smooth functions. We note that if f or g is constant, $c^\pi(f, g) = 0$ holds automatically. Hence, by bilinearity and the assumption of the Lemma, $c^\pi(f, g) = 0$ for polynomial functions of degree ≤ 1. It remains to show that $c^\pi(x_i x_j, x_k x_l) = 0$ for any four indices i, j, k, l. This can be verified, either by direct computation or by verifying that c^π is a biderivation, i.e. a derivation in both its entries. $\square$

Since a vector field being zero is completely determined by it vanishing on a coordinate system, we can verify Equation (1) by testing whether the expression

$$J_{i,j,k} = [\nabla^\pi x_i, \nabla^\pi x_j]x_k - \nabla^\pi\{x_i, x_j\}^\pi x_k$$

[2] I.e. we set $\pi(\alpha, \beta) := \langle \rho(\alpha), \beta \rangle = \beta(\rho(\alpha))$. We can reconstruct ρ from π, because we are in the setting of finite-dimensional vector bundles, $(V^*)^* = V$.

is zero for all i, j, k. By rewriting the commutator of vector fields $[\nabla^\pi x_i, \nabla^\pi x_j]$ as $\nabla^\pi x_i \nabla^\pi x_j - \nabla^\pi x_j \nabla^\pi x_i$ and then replacing all expressions we can by brackets, we obtain:

$$J_{i,j,k} = \{x_i, \{x_j, x_k\}^\pi\}^\pi - \{x_j, \{x_i, x_k\}^\pi\}^\pi - \{\{x_i, x_j\}^\pi, x_k\}^\pi$$

We can calculate

$$\{x_i, \{x_j, x_k\}^\pi\}^\pi = \{x_i, \pi_{j,k}\}^\pi = \sum_l \pi_{i,l} \frac{\partial \pi_{j,k}}{\partial x_l}.$$

Doing the same for the other terms and using skew-symmetry, we obtain

Corollary 1.4. *A family $(\pi_{i,j})_{i,j \in \{1,\dots,n\}}$ of smooth functions defines a Poisson structure on $\mathbb{R}^n$ if and only if it is skew-symmetric ($\pi_{i,j} = -\pi_{j,i}$) and if it satisfies the differential equations*

$$\sum_l \left(\pi_{i,l} \frac{\partial \pi_{j,k}}{\partial x_l} + \pi_{j,l} \frac{\partial \pi_{k,i}}{\partial x_l} + \pi_{k,l} \frac{\partial \pi_{i,j}}{\partial x_l} \right) = 0$$

for all i, j, k.

Example 1.5. *For any skew-symmetric constant $n \times n$-matrix A, $\pi_{i,j} := A_{i,j}$ defines a Poisson structure on $\mathbb{R}^n$.*

Exercise 1.6. *Calculate ∇^π and $\{\cdot,\cdot\}^\pi$ for the Poisson structure on $\mathbb{R}^{2m}$ given by the skew-symmetric matrix*

$$\begin{pmatrix} 0_{m\times m} & -I_m \\ I_m & 0_{m \times m} \end{pmatrix},$$

where I_m is the $m \times m$-identity matrix.

Example 1.7. *In dimension 1, the only Poisson structure is the trivial one $\pi_{1,1} = 0$.*

Exercise 1.8. *When $n = 2$, then a Poisson structure is given by a single function $\pi_{1,2} = -\pi_{2,1} \in C^\infty(\mathbb{R}^2)$. Show that any such function defines a Poisson structure. (It is good if that turns out quite annoying: The reason it might be annoying is that we are missing some mathematical tools, which we are going to learn soon).*

Exercise 1.9. *Try constructing a skew-symmetric π in $\mathbb{R}^3$, which does not define a Poisson structure.*

Exercise 1.10 (Linear Poisson structures)**.** *Let us assume $\pi_{i,j}$ are linear, i.e. can be written as $\pi_{i,j} = \sum_k c^k_{i,j} x_k$. Calculate under which conditions $c^k_{i,j}$ form a Poisson structure.*

Remark 1.11 (Lie algebras)**.** *Let $\mathfrak{g}$ be a finite-dimensional Lie algebra and $\xi_1, ..., \xi_n$ a basis of $\mathfrak{g}$. Then the structure coefficients of $\mathfrak{g}$ with respect to $\xi_1, ..., \xi_n$ are the unique constants $c^k_{i,j}$ satisfying $[\xi_i, \xi_j] = \sum_k c^k_{i,j} \xi_k$. Then the skew-symmetry and Jacobi identity of the Lie algebra recovers exactly the equations from the previous exercise. This is no coincidence - we will see why this is the case later.*

1.1.1 The Poisson structure for classical mechanics

Let us return to the Poisson structure from Exercise 1.6. We think of $M = \mathbb{R}^{2n}$ as phase space with coordinates q_i for positions and p_i for momenta. The Poisson structure satisfies $\{p_i, q_j\} = \delta_{i,j}$ (i.e., 0 for $i \neq j$ and 1 for $i = j$) and $\{p_i, p_j\} = \{q_i, q_j\} = 0$. Let us fix a constant $m > 0$ (the mass of a particle, if you are not a physicist, $m = 1$) and consider a function (Hamiltonian / energy):

$$H(p, q) = \sum_i \frac{p_i^2}{2m} + V(q),$$

where $\sum_i \frac{p_i^2}{2m}$ is the kinetic energy and V is any smooth function of the q variables only (the potential energy). Then $dH = \sum_i \frac{p_i}{m} dp_i + \frac{\partial V}{\partial q_i} dq_i$, and hence

$$\nabla^\pi H = \sum_i \left(\frac{p_i}{m} \partial_{q_i} - \frac{\partial V}{\partial q_i} \partial_{p_i} \right)$$

This means an integral curve $((q(t), p(t)))$ of $\nabla^\pi H$ satisfies the equations of motion of a particle moving through $\mathbb{R}^n$:

$$\dot{q}_i = \frac{p_i}{m}, \qquad \dot{p}_i = -\frac{\partial V}{\partial q_i}(q)$$

or, written in one equation:

$$m\ddot{q}_i = -\frac{\partial V}{\partial q_i}(q).$$

Example 1.12. *For $n = 1$ and $V(q) = \frac{k}{2}q^2$ (k a constant representing the strength of the spring), we obtain the harmonic oscillator $\ddot{q} = \frac{-k}{m}q$.*

Example 1.13. *For $n = 3$ and $V(q) = \frac{-GMm}{|q|}$, (and excluding points with $q = 0$) we obtain the gravitation law, where G is a universal constant and M is the mass of the attracting body. Then the equation is $\ddot{q} = -GM\frac{q}{|q|^3}$.*

So we can reconstruct the equation of motions from the energy using the Poisson structure, but what do we gain? First, we constructed everything so that $\nabla^\pi H(H) = 0$, i.e. H is constant along the trajectories of $\nabla^\pi H$. Secondly, the bracket $\{\cdot,\cdot\}^\pi$ can help us find and understand solutions. If $\{H, f\}^\pi = 0$, then

- The flow of $\nabla^\pi f$ moves solutions of the equations of motion of H to solutions,
- The function f is constant on solutions of the equations of motion.

Exercise 1.14. *Verify these facts.*

1.2 Lie and Poisson algebras (ALGEBRA)

In this section we will summarize what we have seen and introduce the standard notation for it.

Definition 1.15. *For a fiberwise bilinear smooth map $\pi : T^*M \times_M T^*M \to \mathbb{R}$ we write $\pi^\sharp : T^*M \to TM$ for the induced map, uniquely determined by $\pi(\alpha, \beta) = \langle \pi^\sharp\alpha, \beta\rangle = \beta(\pi^\sharp\alpha)$ and call it* anchor *. We call $X_f = \pi^\sharp df$* the Hamiltonian vector field *of f, and $\{f, g\} := X_f g$ the* bracket *of f with g.*

- *When π is skew-symmetric, we call it a* bivector *.*
- *When π satisfies $X_{\{f,g\}} = [X_f, X_g]$ for all $f, g \in C^\infty(M)$ we call it* bracket-preserving *.*
- *When π is skew-symmetric and bracket-preserving we call it a* Poisson bivector *and (M, π) a* Poisson manifold *.*

Remark 1.16. *Usually the words anchor, Hamiltonian vector field and bracket are only used when π is Poisson.*

1.2.1 Lie algebras

In this section we will introduce the algebraic terms for the properties of the bracket $\{\cdot,\cdot\}$, i.e. the notion of (left-) Leibniz algebras and Lie algebras.

Definition 1.17. *A (left) Leibniz algebra structure on a vector space V is a bilinear bracket $[\cdot,\cdot]$ satisfying the following (left) Jacobi identity:*

$$[f, [g, h]] = [[f, g], h] + [g, [f, h]].$$

Bracket-preserving linear maps between Leibniz algebras are called Leibniz algebra homomorphisms.

A Leibniz algebra structure on V is called Lie algebra structure, if it is skew-symmetric. A Lie algebra homomorphism is a Leibniz algebra homomorphism between Lie algebras.

Remark 1.18. *We will use the convention for Leibniz algebras for which $[f,\cdot]$ is a derivation of the bracket. If instead we would write the equation in which $[\cdot, f]$ was a derivation, we would obtain a different notion (called right Leibniz algebras). In the presence of skew-symmetry both notions are equivalent.*

Example 1.19. *The commutator of vector fields is a Lie bracket on $\mathfrak{X}(M)$ for a smooth manifold M.*

Example 1.20. *Let V be a vector space and $W = \mathrm{End}(V)$ be the linear endomorphisms of V. Then the commutator $[A, B] := A \circ B - B \circ A$ is a Lie bracket on W. In particular for $V = \mathbb{R}^n$, we obtain a Lie bracket on the space of square matrices $\mathfrak{gl}(n, \mathbb{R}) := \mathrm{Mat}_{n\times n}(\mathbb{R})$.*

Example 1.21. *When V is a Lie algebra and $\tilde{V} \subset V$ a vector subspace closed under the bracket, then $\tilde{V}$ is a Lie algebra. One can verify that inside $\mathfrak{gl}(n, \mathbb{R})$, the spaces of*

- *trace zero matrices, $\mathfrak{sl}(n, \mathbb{R})$*
- *skew-symmetric matrices $\mathfrak{so}(n, \mathbb{R})$*

are Lie subalgebras.

Exercise 1.22. *Show that:*

- *For any vector space, the zero bracket is a Lie bracket.*
- *For any two Lie algebras $\mathfrak{g}, \mathfrak{g}'$, there exists a unique Lie bracket on $\mathfrak{g} \oplus \mathfrak{g}' = \mathfrak{g} \times \mathfrak{g}'$ such that the inclusions of $\mathfrak{g}, \mathfrak{g}'$ into $\mathfrak{g} \times \mathfrak{g}'$ are Lie algebra homomorphisms and the projections from $\mathfrak{g} \times \mathfrak{g}'$ to $\mathfrak{g}, \mathfrak{g}'$ are Lie algebra homomorphisms.*

The reason we care about Leibniz brackets is the following:

Lemma 1.23. *A $\pi : T^*M \times_M T^*M \to \mathbb{R}$ is bracket-preserving if and only if the induced bracket $\{\cdot, \cdot\} : C^\infty(M) \times C^\infty(M) \to C^\infty(M)$ is a (left) Leibniz bracket.*

Proof. Two vector fields X_1, X_2 are equal if and only if $X_1(h) = X_2(h)$ for any smooth function h. The statement follows by applying $X_{\{f,g\}} = [X_f, X_g]$ to a function h. □

1.2.2 Modules, Derivations and Poisson algebras

In this section we will define the algebraic terms to describe the way how the bracket $\{\cdot, \cdot\}$ interacts with the commutative multiplication of smooth functions. We will phrase things more generally. In this section let A be a commutative associative $\mathbb{R}$-algebra (with unit).

In order to define Poisson algebras, we need to recall a few purely algebraic notions:

Definition 1.24 (Module)**.** *An A-*module *is an abelian group L with a multiplication $\cdot : A \times L \to L$ such that for all elements $a, a' \in A$, $l, l' \in L$:*

$$a \cdot (l + l') = a \cdot l + a \cdot l' \qquad (a + a') \cdot l = a \cdot l + a' \cdot l$$
$$a \cdot (a' \cdot l) = (aa') \cdot l \qquad 1 \cdot l = l.$$

*Let L, L' be A-modules. A map $F : L \to L'$ is called A-*linear *(or an A-*module homomorphism *) if it is a group homomorphism and intertwines the A-multiplications, i.e., $a \cdot (F(l)) = F(a \cdot l)$ for all a, l. We write $\mathrm{Hom}_A(L, L')$ for the space of A-linear maps from L to L'.*

I.e. it is just the definition of a vector space but without requiring the ring to be a field. Note that any A-module is automatically an $\mathbb{R}$-vector space since A contains $\mathbb{R}$ through unitality.

Exercise 1.25. *Show that*

- *A itself is an A-module.*
- *For any two A-modules L, L', the direct sum $L \oplus L' := L \times L'$ has a unique A-module structure making the inclusions and projections A-linear.*

Definition 1.26. *Let L be an A-module. An $\mathbb{R}$-linear map $D : A \to L$ is called a* derivation *if it satisfies the Leibniz rule*

$$D(a \cdot b) = a \cdot D(b) + b \cdot D(a)$$

for all $a, b \in A$. An $\mathbb{R}$-multilinear map $\phi : A^k \to L$ is called a multiderivation *if it is a derivation in each entry. In particular, for $k = 2$ it is then called a* biderivation *.*

Example 1.27. *A vector field $X \in \mathfrak{X}(M)$, seen as a map $C^\infty(M) \to C^\infty(M)$ is a derivation.*

Example 1.28. *For any fiberwise bilinear smooth map $\pi : T^*M \times_M T^*M \to \mathbb{R}$, the bracket $\{\cdot, \cdot\}$ is a biderivation and the map $C^\infty(M) \to \mathfrak{X}(M), f \mapsto X_f$ is a derivation.*

We finally can define:

Definition 1.29. *A* Poisson algebra *is a commutative algebra A with a Lie bracket $\{\cdot,\cdot\}$, which is a biderivation.*

Our considerations immediately imply:

Lemma 1.30. *For a Poisson structure π the corresponding bracket $\{\cdot,\cdot\}$ turns $C^\infty(M)$ into a Poisson algebra.*

We close with the following observation:

Theorem 1.31 ([GM01])**.** *Let A be an algebra without nilpotents*[3] *and $\{\cdot,\cdot\}$ a Leibniz bracket and a biderivation. Then $\{\cdot,\cdot\}$ is skew-symmetric, i.e., a Poisson bracket.*

Proof. We apply the Jacobi identity to $f = x^2/2$, $g = x$, $h = x$ and calculate using the derivation property

$$
\begin{aligned}
&\{x^2/2, \{x, x\}\} = \{\{x^2/2, x\}, x\} + \{x, \{x^2/2, x\}\} \\
\iff &\ldots \\
\iff &x\{x, \{x, x\}\} = x\{\{x, x\}, x\} + x\{x, \{x, x\}\} + 2\{x, x\}^2
\end{aligned}
$$

Subtracting x times the Jacobi identity for $f = g = h = x$, we obtain $2\{x, x\}^2 = 0$, i.e. by the no nilpotents assumption $\{x, x\} = 0$. This is equivalent to skew-symmetry by the polarization formula of linear algebra. ☐

Remark 1.32. *This is a special case of the much more general considerations in [GM01]. An example of this implication failing without the no-nilpotents assumption can also be found there.*

Corollary 1.33. *A bracket-preserving $\pi : T^*M \times_M T^*M \to \mathbb{R}$ is always a Poisson structure (i.e., automatically skew-symmetric).*

1.3 A teaser for non-degeneracy (GEOMETRY)

The zero map induces a trivial Poisson structure whose bracket is the zero bracket. On the other side, the Poisson structure from classical mechanics is fiberwise non-degenerate (as a bilinear map). The primary object of study of this lecture is the latter case. We will express it in terms of the anchor:

Definition 1.34. *A Poisson structure π is called a* symplectic structure *if $\pi^\sharp : T^*M \to TM$ is an injective map. In that case we call π* non-degenerate *, otherwise* degenerate *.*

Remark 1.35. *Since tangent and cotangent bundle are of the same dimension this means that $\pi^\sharp$ is an isomorphism.*

When we were motivating Poisson structures using gradients, we were starting from a metric, a bilinear map $g : TM \times_M TM \to \mathbb{R}$, and constructed the anchor $g^\sharp : T^*M \to TM$ from g. When π is non-degenerate, there is a unique (automatically skew-symmetric) bilinear map $\omega : TM \times_M TM \to \mathbb{R}$ such that $\pi^\sharp = \omega^\sharp$. This form is called the *symplectic form* and to work with it, we will have to introduce Cartan calculus.

[3] i.e., $f^2 = 0$ implies $f = 0$.

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